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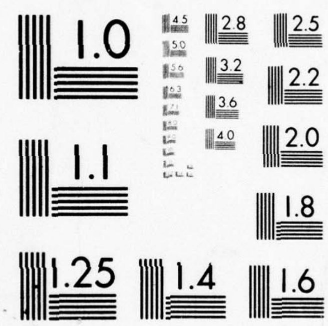

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PROBABILISTIC VALUES FOR GAMES

by

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Robert James Weber

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PROBABILISTIC VALUES FOR GAMES*

by

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1. Introduction

Much attention has been given to methods for measuring the "value" of playing a particular role in an n-person game. The study of various values is motivated by several considerations. One is to determine an equitable distribution of the wealth available to the players through their participation in the game. Another is to help an individual compare his prospects from participation in several games. A study of equitable distributions may shed light upon a player's prospects. However, a study of individual prospects need not yield any information concerning the relative fairness of various distributions of wealth.

The well-known Shapley value assigns to every n-person game an n-vector of payoffs. Since this value serves as a method for determining equitable distributions, it is natural that a defining property of the Shapley value is its "efficiency" (or "Pareto optimality"); that is, the sum of

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*The research reported in this paper was supported, in part, by grant N00014-77-C-0518 from the Office of Naval Research.

the individual payoffs is constrained to equal the payoff achieved through the cooperation of all of the players. However, when the players of a game individually assess their positions in the game, there is no reason to suppose that these assessments (which may depend on subjective or private information) will be jointly efficient. Indeed, conservative assessments may combine into a sub-efficient vector, while optimistic assessments may be super-efficient.

This paper presents an axiomatic development of values for both n-person and infinite (non-atomic) games. Our results will center around the class of "probabilistic" values, which are defined (for finite games) in the next section. Since this class of values includes both the Shapley value and the also-familiar Banzhaf value, our work provides a suitable context for further study of both.

2. Definitions and Notation

For our purposes, we fix a particular set $N = \{1,2,\ldots,n\}$ of players. The collection of coalitions (subsets) in N is denoted by 2^N. A game on N is a real-valued function $v: 2^N \to R$ which assigns a "worth" to each coalition, and which satisfies $v(\emptyset) = 0$. Let $\mathcal{G}$ be the collection of all games on N (note that $\mathcal{G}$ is a $(2^n - 1)$-dimensional vector space), and let v be any game in $\mathcal{G}$. The game v is monotonic if $v(S) \geq v(T)$ for all $S \supset T$; v is superadditive if $v(S \cup T) \geq v(S) + v(T)$ whenever $S \cap T = \emptyset$. The class of all monotonic games is denoted by $\mathcal{M}$, and the class of all superadditive games by $\mathcal{S}$. For future reference, note that $\mathcal{M}$ and $\mathcal{S}$ are cones in $\mathcal{G}$; that is, each is closed under addition, and under multiplication by nonnegative real numbers. Also note that neither class contains the other.

If the game v takes only the values 0 and 1, then v is <u>simple</u>. If $v(S) = 1$, then S is a <u>winning coalition</u>; otherwise S is a <u>losing coalition</u>. $\mathcal{G}^*$, $\mathcal{M}^*$, and $\mathcal{S}^*$ denote, respectively, the class of all simple games on N, those which are monotonic, and those which are superadditive. For simple games, note that superadditivity implies monotonicity; hence, $\mathcal{M}^* \supset \mathcal{S}^*$. (Some authors prefer to restrict the term "simple game" to elements of $\mathcal{M}^*$; the more general games $\mathcal{G}^*$ are then called "0-1 games.")

Two special types of games will play an important role in our work. For any nonempty coalition T, let v_T be defined by $v_T(S) = 1$ if $S \supset T$, and 0 otherwise. Also, let $\hat{v}_T$ be defined by $\hat{v}_T(S) = 1$ if $S \supsetneq T$, and 0 otherwise. Let $\mathcal{C} = \{v_T : \emptyset \neq T \subset N\}$, and $\hat{\mathcal{C}} = \{\hat{v}_T : \emptyset \neq T \subset N\}$; any game in $\mathcal{C}$ is a <u>carrier game</u>. Observe that every game in $\mathcal{C}$ or $\hat{\mathcal{C}}$ is monotonic, superadditive, and simple. We shall occasionally refer to the game $\hat{v}_\emptyset$ defined by $\hat{v}_\emptyset(S) = 1$ for all nonempty coalitions S. This game is monotonic and simple, but is not superadditive.

For any collection $\mathcal{T} \subset \mathcal{G}$ of games, and for any player $i \in N$, a <u>value</u> for i on $\mathcal{T}$ is a function $\phi_i : \mathcal{T} \to R$. As we have previously indicated, the value $\phi_i(v)$ of a particular game v represents an assessment by i of his prospects from playing the game. This definition stands somewhat in contrast to the more traditional definition of a "group value" $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ which associates an n-vector with each game. The construction of group values from our individual values will be treated later in this paper.

Recently, Blair [1] and Dubey [3] have discussed a family of values which arise from individual perceptions of the coalition-formation process.

(Earlier discussions of related matters appear in [4] and [5].) Fix a player i, and let $\{p_T^i: T \subset N\backslash i\}$ be a probability distribution over the collection of coalitions not containing i. (Incidentally, notice that we shall often omit the braces when writing one-player coalitions such as $\{i\}$.) A value ϕ_i for i on $\mathcal{J}$ is a <u>probabilistic value</u> if, for every $v \in \mathcal{J}$,

$$\phi_i(v) = \sum_{T \subset N\backslash i} p_T^i [v(T \cup i) - v(T)] \quad .$$

Let i view his participation in a game as consisting merely of joining some coalition S, and then receiving as a reward his marginal contribution $v(S \cup i) - v(S)$ to the coalition. If, for each $T \subset N\backslash i$, p_T^i is the (subjective) probability that he joins coalition T, then $\phi_i(v)$ is simply his expected payoff from the game.

Both the Shapley and Banzhaf values are instances of probabilistic values. The Banzhaf value (for an individual player i) arises from the subjective belief that the player is equally likely to join any coalition; that is, $p_T^i = 1/(2^{n-1})$ for all $T \subset N\backslash i$. The Shapley value arises from the belief that the coalition he joins is equally likely to be of any size t $(0 \leq t \leq n-1)$, and that all coalitions of size t are equally likely;

that is, $p_T^i = \frac{1}{n} \cdot \frac{1}{\binom{n-1}{t}} = \frac{t! \cdot (n-t-1)!}{n!}$ for all $T \subset N\backslash i$, where

$t = |T|$.

In the following sections, we shall investigate several reasonable conditions which a value might be expected to satisfy. We will find that the only values which satisfy these conditions are closely related to the probabilistic values.

3. The Linearity and Dummy Axioms

Given a game v, and any constant $c > 0$, consider the game cv defined by $(cv)(S) = c \cdot v(S)$ for all $S \subset N$. It seems reasonable to assume that such a rescaling of the original game would simply rescale a player's assessment of his prospects from playing the game. Similarly, let v and w be games, and consider the game $v + w$ defined by $(v + w)(S) = v(S) + w(S)$ for all $S \subset N$. A rational player, facing the latter game, might well consider his prospective gain to be the sum of his prospective gains from the two original games.

Consider a cone $\mathcal{T}$ of games in $\mathcal{G}$. A <u>linear function</u> on $\mathcal{T}$ is a function $f: \mathcal{T} \to R$ satisfying $f(v + w) = f(v) + f(w)$ and $f(cv) = c \cdot f(v)$ for all $v, w \in \mathcal{T}$ and $c > 0$. Let ϕ_i be a value for i on $\mathcal{T}$. The preceding comments are reflected in the following criterion.

<u>Linearity Axiom</u>. ϕ_i is a linear function on $\mathcal{T}$.

Since $\mathcal{G}$, $\mathcal{M}$, and $\mathcal{S}$ are all cones in $\mathcal{G}$, the following theorem applies to a value on any of these domains.

<u>THEOREM 1</u>. Let ϕ_i be a value for i on a cone $\mathcal{T}$ of games. Assume that ϕ_i satisfies the linearity axiom. Then there is a collection of constants $\{a_T : T \subset N\}$ such that for all $v \in \mathcal{T}$,

$$\phi_i(v) = \sum_{T \subset N} a_T v(T) .$$

Proof. ϕ_i has a unique linear extension to the linear subspace $\mathcal{L} \subset \mathcal{G}$ spanned by $\mathcal{J}$. This extension can in turn be extended to a linear function ϕ_i^{ext} on all of $\mathcal{G}$, by defining ϕ_i^{ext} arbitrarily on a basis of the orthogonal complement of $\mathcal{L}$.

For any nonempty $T \subset N$, define the game w_T by $w_T(S) = 1$ if $S = T$, and 0 otherwise. Then $\{w_T : \emptyset \neq T \subset N\}$ is a basis for $\mathcal{G}$, and ϕ_i^{ext} is uniquely determined by its values on this basis. Any $v \in \mathcal{G}$ can be written as $v = \sum_{\emptyset \neq T \subset N} v(T) \cdot w_T$; since ϕ_i^{ext} is linear,

$$\phi_i^{ext}(v) = \sum_{\emptyset \neq T \subset N} v(T) \cdot \phi_i^{ext}(w_T) \quad .$$

However, ϕ_i is simply the restriction of ϕ_i^{ext} to $\mathcal{J}$. Therefore, upon taking $a_T = \phi_i^{ext}(w_T)$ for all nonempty $T \subset N$, and defining $a_\emptyset$ arbitrarily, we obtain the desired result. $\square$

A player i is a dummy in the game v if $v(S \cup i) = v(S) + v(i)$ for every $S \subset N \backslash i$. This terminology derives from the observation that such a player has no meaningful strategic role in the game; no matter what the situation, he contributes precisely $v(i)$. Therefore, the following criterion seems reasonable. Let ϕ_i be a value for i on a collection $\mathcal{J}$ of games.

Dummy Axiom. If i is a dummy in $v \in \mathcal{J}$, then $\phi_i(v) = v(i)$.

This axiom actually has two aspects. While specifying the prospective gain of a dummy in a game v, it implicitly states that ϕ_i and v are measured in common units, under a common normalization. These aspects are exploited separately in the proof of the following result. Recall that $\mathcal{C}$ denotes the collection of carrier games.

<u>*THEOREM 2*</u>. *Let* ϕ_i *be a value for* i *on a collection* $\mathcal{G}$ *of games, defined by* $\phi_i(v) = \sum_{T \subset N} a_T v(T)$ *for every* $v \in \mathcal{G}$. *Assume that* $\mathcal{G}$ *contains* $\mathcal{C}$.

Then there is a collection of constants $\{p_T : T \subset N\backslash i\}$ *satisfying* $\sum_{T \subset N\backslash i} p_T = 1$,

such that for every $v \in \mathcal{G}$,

$$\phi_i(v) = \sum_{T \subset N/i} p_T[v(T \cup i) - v(T)] \quad .$$

<u>Proof</u>. First, note that for any nonempty $T \subset N\backslash i$, player i is a dummy in $v_T \in \mathcal{C}$. Therefore, $\phi_i(v_T) = v_T(i) = 0$. It follows that $\phi_i(v_{N\backslash i}) = a_N + a_{N\backslash i} = 0$. For inductive purposes, assume it has been shown that $a_{T \cup i} + a_T = 0$ for every $T \subset N\backslash i$ with $|T| \geqq k \geqq 2$. (The case $k = n - 1$ has just been established.) Take any fixed $S \subset N\backslash i$ with $|S| = k - 1$. Then

$$\phi_i(v_S) = \sum_{T \supset S} a_T = \left\{ \sum_{\substack{T \subset N\backslash i \\ T \supsetneq S}} (a_{T \cup i} + a_T) \right\} + (a_{S \cup i} + a_S)$$

$$= a_{S \cup i} + a_S = 0 \quad ;$$

the next-to-last equality follows from the induction hypothesis, and the last from the dummy axiom.

Therefore, $a_{T \cup i} + a_T = 0$ for all $T \subset N\backslash i$ with $0 < |T| \leqq n - 1$. For every such T , define $p_T = a_{T \cup i} = -a_T$. Also, define $p_\emptyset = a_i$. Then for every $v \in \mathcal{G}$,

$$\phi_i(v) = \sum_{T \subset N} a_T v(T) = \sum_{T \subset N\backslash i} p_T[v(T \cup i) - v(T)] \quad .$$

Consider $v_i \in \mathcal{C}$. Player i is a dummy in this game; indeed, every player is a dummy in v_i. Therefore, $\phi_i(v_i) = v_i(i) = 1$. But, since $v_i(T \cup i) - v_i(T) = 1$ for every $T \subset N\backslash i$, the expression in the preceding paragraph yields $\phi_i(v_i) = \sum_{T\subset N\backslash i} p_T$. □

When this theorem is taken in conjunction with the preceding one, we obtain the following result.

<u>THEOREM 3</u>. *Let ϕ_i be a value for i on $\mathcal{G}$, $\mathcal{M}$, or $\mathcal{S}$. Assume that ϕ_i satisfies the linearity and dummy axioms. Then there is a collection of constants $\{p_T: T \subset N\backslash i\}$ satisfying $\sum_{T\subset N\backslash i} p_T = 1$, such that for every game v in the domain of ϕ_i,*

$$\phi_i(v) = \sum_{T\subset N\backslash i} p_T[v(T \cup i) - v(T)] .$$

4. <u>The Monotonicity Axiom</u>

Let v be any monotonic game. A player i, facing the prospect of playing this game, may be uncertain concerning his eventual payoff. However, for every $T \subset N\backslash i$, $v(T \cup i) - v(T) \geq 0$; therefore player i knows, at the least, that his presence will never "hurt" a coalition. This motivates the following criterion. Let ϕ_i be a value for i on a collection $\mathcal{J}$ of games.

<u>*Monotonicity Axiom*</u>. *If $v \in \mathcal{J}$ is monotonic, then $\phi_i(v) \geq 0$.*

The following proposition will be of value.

<u>Proposition</u>. *Let* ϕ_i *be a value for* i *on a collection* $\mathcal{J}$ *of games. Assume that there is a collection of constants* $\{p_T: T \subset N\backslash i\}$ *, such that for all* $v \in \mathcal{J}$*,*

$$\phi_i(v) = \sum_{T \subset N\backslash i} p_T[v(T \cup i) - v(T)] \quad .$$

Further assume that $\mathcal{J}$ *contains the game* $\hat{v}_T$ *, for some* $T \subset N\backslash i$ *(note that* T *may be empty), and assume that* ϕ_i *satisfies the monotonicity axiom. Then* $p_T \geqq 0$.

<u>Proof</u>. The game $\hat{v}_T$ is monotonic. Therefore, $\phi_i(\hat{v}_T) = p_T \geqq 0$. □

The collections of games $\mathcal{G}$ and $\mathcal{M}$ each contain $\hat{\mathcal{C}}$, and also contain $\hat{v}_\emptyset$. On the other hand, $\mathcal{S}$ contains $\hat{\mathcal{C}}$, but not $\hat{v}_\emptyset$. Therefore, we have the following theorems.

<u>*THEOREM 4*</u>. *Let* ϕ_i *be a value for* i *on* $\mathcal{G}$ *or* $\mathcal{M}$*. Assume that* ϕ_i *satisfies the linearity, dummy, and monotonicity axioms. Then* ϕ_i *is a probabilistic value. Furthermore, every probabilistic value on* $\mathcal{G}$ *or* $\mathcal{M}$ *satisfies these three axioms.*

<u>*THEOREM 5*</u>. *Let* ϕ_i *be a value for* i *on* $\mathcal{S}$ *. Assume that* ϕ_i *satisfies the linearity, dummy, and monotonicity axioms. Then there is a collection of constants* $\{p_T: T \subset N\backslash i\}$ *satisfying* $\sum_{T \subset N\backslash i} p_T = 1$ *, and* $p_T \geqq 0$ *for all nonempty* $T \subset N\backslash i$ *, such that for every game* $v \in \mathcal{S}$*,*

$$\phi_i(v) = \sum_{T\subset N\backslash i} p_T[v(T \cup i) - v(T)] \quad .$$

Furthermore, every such value on $\mathcal{S}$ satisfies these three axioms.

In the case of values on $\mathcal{G}$ or $\mathcal{M}$, we thus have a natural axiomatic characterization of the probabilistic values. However, for values on $\mathcal{S}$ we are unable to rule out the possibility that $p_\emptyset < 0$. This phenomenon is investigated in the next section.

5. Values for Superadditive Games

It is natural to seek an explanation of the preceding results. A value for a class of games yields a relative evaluation of one's prospects from playing the various games. If the class of games is sufficiently rich, the only evaluation functions satisfying certain reasonable criteria are the probabilistic values. Why, if one's consideration is restricted solely to superadditive games, does the class of reasonable evaluation functions broaden in the indicated manner? We shall attempt to provide a rationale.

Consider any particular game v. A player i, faced with the prospect of playing this game, may seek to determine the amount of gain which he is "guaranteed," in the sense that he contributes at least this amount to any coalition which he joins. In the case where v is superadditive, this "floor" to his expectation is precisely $v(i)$, since $v(T \cup i) - v(T) \geq v(i)$ for all $T \subset N\backslash i$ (and since, when $T = \emptyset$, his marginal contribution is exactly $v(i)$). Taking this amount as assured, the player will then strive to achieve as great a reward as he can, in the new game $v^{(i)}$ defined by

$$v^{(i)}(S) = \begin{cases} v(S) & \text{if } i \notin S , \\ v(S) - v(i) & \text{otherwise} . \end{cases}$$

(This is the game that he perceives himself to be playing, after having mentally "withdrawn" the amount $v(i)$ from the game.) However, any gain from this new game is uncertain, and depends upon such factors as the bargaining ability of the player. Hence, the two amounts under consideration, $v(i)$ and his gain from playing $v^{(i)}$, are measured respectively in "certain" and "uncertain" units.

Assume that the player's attitude toward risk is such that one unit of uncertain gain is worth γ units of certain gain to him. (Hence, $\gamma < 1$ corresponds to risk-aversion, and $\gamma = 1$ to risk-neutrality.) Further assume that he evaluates his prospects, from any game v with $v(i) = 0$, in terms of a probabilistic value $\phi_i(v)$. Then, his evaluation of any superadditive game v, expressed in units of certain gain, will be

$$\xi_i(v) = \gamma \cdot \phi_i(v^{(i)}) + v(i) .$$

One would expect an aversion to risk to limit a player's options. That such is the case is the impact of the following theorem. Let P be the set of probabilistic values on $\mathcal{S}$, and for any $\gamma \geq 0$ let $V(\gamma) = \{\xi_i : \xi_i$ is a value on $\mathcal{S}$, and for some $\phi_i \in P$, $\xi_i(v) = \gamma \cdot \phi_i(v^{(i)}) + v(i)$ for all $v \in \mathcal{S}\}$. This is the set of all evaluation functions on $\mathcal{S}$ arising from the considerations discussed previously, when γ represents player i's attitude toward uncertain gain.

<u>*THEOREM 6*</u>. *If* $0 \leq \gamma' < \gamma$, *then* $V(\gamma') \subsetneq V(\gamma)$. *Furthermore,* $V(1) = P$.

<u>Proof</u>. If $0 \leqq \gamma' < \gamma$, then any $\xi_i \in V(\gamma')$ corresponds to some $\phi'_i \in P$, which is in turn associated with a probability distribution $\{p_T\colon T \subset N\backslash i\}$. But then, let $\phi_i \in P$ be associated with the probability distribution $\{q_T\colon T \subset N\backslash i\}$, where $q_T = \frac{\gamma'}{\gamma} \cdot p_T$ for all nonempty $T \subset N\backslash i$, and $q_\emptyset = 1 - \sum_{T \neq \emptyset} q_T$. It follows that $\xi_i(v) = \gamma \cdot \phi_i(v^{(i)}) + v(i)$ for all $v \in \mathcal{S}$, so $\xi_i \in V(\gamma)$. Hence, $V(\gamma') \subset V(\gamma)$.

Consider any probability distribution $\{p_T\colon T \subset N\backslash i\}$ such that $p_\emptyset = 0$. Then, if ϕ_i is the associated probabilistic value on $\mathcal{S}$, $\xi_i(v) = \gamma \cdot \phi_i(v^{(i)}) + v(i)$ defines a value $\xi_i \in V(\gamma)$ which is not in $V(\gamma')$ for any $\gamma' < \gamma$. Hence the indicated containment is strict.

Finally, observe that, when $\gamma = 1$, every value ξ_i in $V(\gamma) = V(1)$ is of the form

$$\begin{aligned}
\xi_i(v) &= \phi_i(v^{(i)}) + v(i) \\
&= \left\{ \sum_{T \subset N\backslash i} p_T[(v(T \cup i) - v(i) - v(T)]\right\} + v(i) \\
&= \sum_{T \subset N\backslash i} p_T[v(T \cup i) - v(T)] \\
&= \phi_i(v) \quad ,
\end{aligned}$$

so $V(1) = P$. □

Another point of view is offered by this theorem. If a player wishes to evaluate his prospects from superadditive games, he can satisfy our criteria of rationality while still basing his evaluation in part on his posture toward risk. However, these same criteria, when applied to the evaluation of broader classes of games, force the player into a posture of risk-neutrality. It would be of interest to learn precisely from where this consequence of risk-neutrality arises.

6. Values for Simple Games

Simple games, particularly those which are monotonic, are often used to represent political games. A value for a player may then indicate the player's perceived political power in various games. Under this interpretation, the dummy and monotonicity axioms remain reasonable. However, the linearity axiom does not seem to apply; indeed, the sum of simple games is generally not simple.

An alternative axiom has been suggested by Dubey [2] . For any games v and w, define $v \vee w$ by $(v \vee w)(S) = \max (v(S), w(S))$ and define $v \wedge w$ by $(v \wedge w)(S) = \min (v(S), w(S))$, for all $S \subset N$. If v and w are simple, then $v \vee w$ and $v \wedge w$ are also simple. A coalition is winning in $v \vee w$ if it wins in either v or w; it is winning in $v \wedge w$ if it wins in both. Therefore, each coalition wins as often in v and w together as it does in $v \vee w$ and $v \wedge w$ together. Let ϕ_i be a value for i on a collection $\mathcal{J}$ of games.

Transfer Axiom. *If* v, w, $v \vee w$, and $v \wedge w$ *are all in* $\mathcal{J}$, *then* $\phi_i(v) + \phi_i(w) = \phi_i(v \vee w) + \phi_i(v \wedge w)$.

The name of this axiom is motivated by the following observation. The game $v \wedge w$ arises from v when all of the coalitions which win only in v are made losing; $v \vee w$ arises from w when these same coalitions are made winning. Hence, $v \wedge w$ and $v \vee w$ arise from v and w when winning coalitions are "transferred" from one game to the other.

We require several definitions. Let v be a simple game. A minimal winning coalition in v is a winning coalition with no proper subsets which are also winning; a hole in v is a losing coalition with a winning subset. Note that the monotonic simple games are precisely those without holes.

Let $\mathcal{J}$ be a collection of simple games, and let v be any game in $\mathcal{J}$. We define two types of operations which can be performed on v. Let T be a minimal winning coalition in v. Define the game v^{-T} by $v^{-T}(S) = v(S)$ for all $S \neq T$, with $v^{-T}(T) = 0$; v^{-T} arises from v by the deletion of a minimal winning coalition. On the other hand, let T be a hole in v, and define the game v^{+T} by $v^{+T}(S) = v(S)$ for all $S \neq T$, with $v^{+T}(T) = 1$; v^{+T} arises from v by the insertion of a (new) winning coalition. The collection $\mathcal{J}$ is closed under deletion and insertion if these operations, applied to any game in $\mathcal{J}$, give rise only to other games in $\mathcal{J}$. In particular, $\mathcal{G}^*$, $\mathcal{M}^*$, and $\mathcal{S}^*$ are all closed under deletion and insertion.

The following result is an analogue of Theorem 1.

THEOREM 7. Let $\mathcal{J}$ be a collection of simple games which contains $\mathcal{C}$ and is closed under deletion and insertion. Let ϕ_i be a value for i on $\mathcal{J}$, and assume that $\phi_i(\hat{v}_N) = 0$. Finally, assume that ϕ_i satisfies the*

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*Recall that the game $\hat{v}_N$ is defined by $\hat{v}_N(S) = 0$ for all $S \subset N$. This game is contained in every nonempty collection of games which is closed under deletion, and every player in N is a dummy in the game.

transfer axiom. Then there is a collection of constants $\{a_T: T \subset N\}$ *such that, for all games* $v \in \mathcal{J}$,

$$\phi_i(v) = \sum_{T \subset N} a_T v(T) \quad .$$

Proof. We claim that ϕ_i is determined on all of $\mathcal{J}$ by its values on $\mathcal{C}$. In order to verify this claim, first consider the collection $\mathcal{J}_M$ of monotonic games in $\mathcal{J}$. This subcollection of $\mathcal{J}$ is also closed under deletion and insertion, and contains $\mathcal{C}$. Since $v_N \in \mathcal{C}$, the claim is trivially true for this game. Assume that the claim has been verified for all games in $\mathcal{J}_M$ which have at most k winning coalitions (the only game in $\mathcal{J}_M$ with just one winning coalition is v_N), and let $v \in \mathcal{J}$ be any game with $k + 1$ winning coalitions. Let T be any minimal winning coalition in v , and consider the games v_T , v^{-T} , and $v_T \wedge v^{-T}$. The first is a carrier game, while the latter two are both in $\mathcal{J}_M$ and have no more than k winning coalitions. Since $v_T \vee v^{-T} = v$, we have from the transfer axiom that $\phi_i(v) = \phi_i(v_T) + \phi_i(v^{-T}) - \phi_i(v_T \wedge v^{-T})$. It follows from the induction hypothesis that $\phi_i(v)$ depends only on the values of ϕ_i on $\mathcal{C}$. This verifies the claim throughout $\mathcal{J}_M$. (Observe that the game $\hat{v}_N$ requires special treatment; since it has no winning coalitions, it is not covered by the induction.)

Next, assume that the claim holds for all games in $\mathcal{J}$ which have at most k holes (the case $k = 0$ has just been treated) , and let $v \in \mathcal{J}$ be a game with $k + 1$ holes. Let T be any hole of maximum cardinality, and consider the games v_T , $v \wedge v_T = \hat{v}_T$, and $v \vee v_T = v^{+T}$. The first of these is in $\mathcal{C}$, the second is in $\mathcal{J}_M$, and the third is in $\mathcal{J}$

and has only k holes. Since $\phi_i(v) = \phi_i(v \vee v_T) + \phi_i(v \wedge v_T) - \phi_i(v_T)$, it follows (by induction) that $\phi_i(v)$ depends only on the values of ϕ_i on $\mathcal{C}$. This completes the verification of the claim.

We have just seen that ϕ_i is determined by its values on $\mathcal{C}$. Since* $\mathcal{C}$ is a basis for $\mathcal{G}$, there is a unique linear function ϕ_i^{lin} on $\mathcal{G}$ which coincides with ϕ_i on $\mathcal{C}$. This linear function must satisfy the transfer axiom, because $(v \vee w) + (v \wedge w) = v + w$ for all v and w in $\mathcal{G}$. Therefore, ϕ_i^{lin} and ϕ_i must coincide on $\mathcal{T}$. Since ϕ_i^{lin} can be expressed in terms of its values on the basis $\{w_T\colon \emptyset \neq T \subset N\}$ of $\mathcal{G}$ (see the proof of Theorem 1), it follows that ϕ_i has the desired form. □

We can now invoke Theorem 2 and the proposition concerning monotonicity, in order to obtain analogues of Theorems 4 and 5.

<u>*THEOREM 8*</u>. *Let ϕ_i be a value for i on $\mathcal{G}^*$ or $\mathcal{M}^*$. Assume that ϕ_i satisfies the transfer, dummy, and monotonicity axioms. Then ϕ_i is a probabilistic value. Furthermore, every probabilistic value on $\mathcal{G}^*$ or $\mathcal{M}^*$ satisfies these three axioms.*

<u>*THEOREM 9*</u>. *Let ϕ_i be a value for i on $\mathcal{S}^*$. Assume that ϕ_i satisfies the transfer, dummy, and monotonicity axioms. Then there is a collection of constants $\{p_T\colon T \subset N\backslash i\}$ satisfying $\sum_{T \subset N\backslash i} p_T = 1$, and*

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*Assume that $\sum_{\emptyset \neq S \subset N} c_S v_S = 0$. Then for any nonempty $T \subset N$, $\sum_{\emptyset \neq S \subset T} c_S = 0$. Solving this system of equations successively for $|T| = 1,2,\ldots,n$ yields $c_T = 0$ for all $T \subset N$. Hence the $2^n - 1$ games v_T are linearly independent in $\mathcal{G}$.

$p_T \geq 0$ *for all nonempty* $T \subset N\backslash i$ *, such that for every game* $v \in \mathcal{S}^*$ *,*

$$\phi_i(v) = \sum_{T \subset N\backslash i} p_T[v(T \cup i) - v(T)] \quad .$$

Furthermore, every such value on $\mathcal{S}^*$ *satisfies these three axioms.*

The discussion of the previous section, interpreting the class of values on $\mathcal{S}$, applies with equal strength to $\mathcal{S}^*$.

7. Symmetric Probabilistic Values

A probabilistic value assesses the relative desirability of being a particular player in various games. At times, one might also want to compare the desirability of playing various roles within a particular game. Such comparisons can be facilitated by the use of a collection $\phi = (\phi_1,\ldots,\phi_n)$ of values, with $\phi_i(v)$ representing the value of being player i in game v . Such a collection is a <u>group value</u>.

Let $\pi = (\pi(1),\ldots,\pi(n))$ be any permutation of N . For any $S \subset N$, define $\pi S = \{\pi(i): i \in S\}$. The game πv is defined by $(\pi v)(\pi S) = v(S)$ for all $S \subset N$. (πv arises upon the re-labelling of the players $1,\ldots,n$ with the labels $\pi(1),\ldots,\pi(n)$.) Let $\mathcal{J}$ be a collection of games with the property that, if $v \in \mathcal{J}$, then every $\pi v \in \mathcal{J}$; such a collection is <u>symmetric</u>.

Let $\phi = (\phi_1,\ldots,\phi_n)$ be a group value on $\mathcal{J}$. For the comparison of roles in a game to be meaningful, the evaluation of a particular position should depend on the structure of the game, but not on the labels of the players.

Symmetry Axiom. *For every* $v \in \mathcal{J}$ *and every permutation* π *of* N *, and for every* $i \in N$ *,* $\phi_i(v) = \phi_{\pi(i)}(\pi v)$.

Observe that each of the classes $\mathcal{G}, \mathcal{M}, \mathcal{S}, \mathcal{G}^*, \mathcal{M}^*$, and $\mathcal{S}^*$ contains both $\mathcal{C}$ and $\hat{\mathcal{C}}$; also each of these classes is symmetric. Therefore, the following theorem applies to values on any of these classes.

THEOREM 10. *Let* $\mathcal{J}$ *be a symmetric collection of games, containing* $\mathcal{C}$ *and* $\hat{\mathcal{C}}$. *Let* $\phi = (\phi_1, \ldots, \phi_n)$ *be a group value on* $\mathcal{J}$ *, such that for each* $i \in N$ *and* $v \in \mathcal{J}$ *,*

$$\phi_i(v) = \sum_{T \subset N \setminus i} p_T^i [v(T \cup i) - v(T)] \quad .$$

Assume that ϕ *satisfies the symmetry axiom. Then there are constants* $\{p_t\}_{t=0}^{n-1}$ *such that for all* $i \in N$ *and* $T \subset N \setminus i$ *,* $p_T^i = p_{|T|}$.

Proof. For any $i \in N$, let T_1 and T_2 be any two coalitions in $N \setminus i$ satisfying $0 < |T_1| = |T_2| < n - 1$. Consider a permutation π of N , which takes T_1 into T_2 while leaving i fixed. Then $p_{T_1}^i = \phi_i(\hat{v}_{T_1}) = \phi_i(\hat{v}_{T_2}) = p_{T_2}^i$, where the central equality is a consequence of the symmetry axiom.

Next, let i and j be distinct players in N , and let T be a nonempty coalition in $N \setminus \{i,j\}$. Consider the permutation π which interchanges i and j while leaving the remaining players fixed. Then $\pi \hat{v}_T = \hat{v}_T$, and $p_T^i = \phi_i(\hat{v}_T) = \phi_j(\hat{v}_T) = p_T^j$, where the central equality is again a consequence of the symmetry axiom. Combining this with the previous result, we find that for every $0 < t < n-1$ there is a p_t such that $p_T^i = p_t$ for every $i \in N$ and $T \subset N\backslash i$ with $|T| = t$.

Again, for distinct players i and j, let π interchange i and j while leaving the remaining players fixed. Then $p^i_{N\backslash i} = \phi_i(v_N) = \phi_j(v_N) = p^j_{N\backslash j}$. Let p_{n-1} be this common value. Then for all $i \in N$, $p^i_{N\backslash i} = p_{n-1}$.

Finally, for each $i \in N$,

$$p^i_\emptyset = 1 - \sum_{\substack{T\subset N\backslash i \\ T\neq\emptyset}} p^i_T = 1 - \sum_{t=1}^{n-1} \binom{n-1}{t} p_t \quad ;$$

this last expression is independent of i.

Therefore, $p^i_\emptyset = p^j_\emptyset$ for all $i,j \in N$. Letting p_0 be this common value completes the proof of the theorem. □

We shall return to this result later in the paper, when we briefly consider the Shapley value.

8. Efficiency without Symmetry: Random-order Values

Consider a collection $\phi = (\phi_1,\ldots,\phi_n)$ of values, all on the domain $\mathcal{J}$, one for each player in N. Depending on the game v under consideration, the players' assessments, as a group, of their individual prospects may be either optimistic or pessimistic; that is, $\sum_{i\in N} \phi_i(v)$ may be either greater than or less than $v(N)$. However, if the group assessment is neither optimistic nor pessimistic, the payoff vector $\phi(v) = (\phi_1(v),\ldots,\phi_n(v))$ may be taken as an equitable distribution of the resources available to the grand coalition N. Therefore, it is of interest to study those collections of values $\phi = (\phi_1,\ldots,\phi_n)$ which meet the following criterion.

<u>*Efficiency Axiom*</u>. *For every* $v \in \mathcal{J}$, $\sum_{i \in N} \phi_i(v) = v(N)$.

A group value satisfying this axiom is said to be <u>efficient</u>.

Any efficient group value ϕ provides a fair distribution scheme for the games in $\mathcal{J}$. The following theorem characterizes all such group values.

<u>*THEOREM 11*</u>. *Let* $\phi = (\phi_1, \ldots, \phi_n)$ *be a group value on* $\mathcal{J}$ *, defined for all* $i \in N$ *and all* $v \in \mathcal{J}$ *by* $\phi_i(v) = \sum_{T \subset N \setminus i} p_T^i[v(T \cup i) - v(T)]$. *Assume that* $\mathcal{J}$ *contains* $\mathcal{C}$ *and* $\hat{\mathcal{C}}$. *Then* ϕ *satisfies the efficiency axiom if and only if* $\sum_{i \in N} p_{N \setminus i}^i = 1$, *and* $\sum_{i \in T} p_{T \setminus i}^i = \sum_{j \notin T} p_T^j$ *for every nonempty* $T \subsetneq N$.

<u>Proof</u>. For any $v \in \mathcal{G}$, let $\phi_N(v) = \sum_{i \in N} \phi_i(v)$. Then

$$\begin{aligned}\phi_N(v) &= \sum_{i \in N} \sum_{T \subset N \setminus i} p_T^i[v(T \cup i) - v(T)] \\ &= \sum_{T \subset N} v(T)\left(\sum_{i \in T} p_{T \setminus i}^i - \sum_{j \notin T} p_T^j\right) .\end{aligned}$$

It is immediately clear that any ϕ which satisfies the conditions of the theorem is efficient; that is, $\phi_N(v) = v(N)$.

For any nonempty $T \subset N$, consider the games v_T and $\hat{v}_T$. Since $v_T(S) = \hat{v}_T(S)$ for all $S \neq T$, and $v_T(T) = 1$ while $\hat{v}_T(T) = 0$, it follows from the preceding equation that

$$\phi_N(v_T) - \phi_N(\hat{v}_T) = \sum_{i \in T} p_{T \setminus i}^i - \sum_{j \notin T} p_T^j .$$

However, $v_T(N) - \hat{v}_T(N)$ is 1 if $T = N$, and is 0 otherwise. Therefore, if ϕ satisfies the efficiency axiom, then the indicated conditions must also hold. □

It is conceivable that the efficiency of a group value is an artifact, existing in spite of the fact that the players have grossly different views of the world. However, we can define a family of group values, each of which arises from a viewpoint common to all of the players. Let $\{r_\pi : \pi \in \Pi\}$ be a probability distribution over the set Π of $n!$ orderings of N; r_π is the probability associated with the ordering $\pi = (i_1, \ldots, i_n)$ in which the k-th player is player i_k. For any ordering $\pi = (i_1, \ldots, i_n)$, let $\pi^{i_k} = \{i_1, \ldots, i_{k-1}\}$ be the set of predecessors of i_k in π. A <u>random-order group value</u> $\xi = (\xi_1, \ldots, \xi_n)$ on $\mathcal{J}$ is defined by

$$\xi_i(v) = \sum_{\pi \in \Pi} r_\pi [v(\pi^i \cup i) - v(\pi^i)] \quad ,$$

for all $i \in N$ and all $v \in \mathcal{J}$.

An interpretation of this definition can be given. Assume that the players have as their goal the eventual formation of the grand coalition, N. Further assume that they see coalition-formation as a sequential process: given any ordering π of the players, each player i joins with his predecessors in π, making the marginal contribution $v(\pi^i \cup i) - v(\pi^i)$ in the game v. Then, if the players share a common perception $\{r_\pi : \pi \in \Pi\}$ of the likelihood of the various orderings, the expected marginal contribution of a player is precisely his component of the random-order group value.

<u>THEOREM 12</u>. *Let* $\xi = (\xi_1,\ldots,\xi_n)$ *be a random-order group value on* $\mathcal{J}$ *, associated with with the probability distribution* $\{r_\pi : \pi \in \Pi\}$ *. There exists a collection* $\phi = (\phi_1,\ldots,\phi_n)$ *of probabilistic values on* $\mathcal{J}$ *, such that* $\phi_i(v) = \xi_i(v)$ *for all* $i \in N$ *and all* $v \in \mathcal{J}$ *. Furthermore,* ϕ *satisfies the efficiency axiom.*

<u>Proof</u>. For any $i \in N$ and $v \in \mathcal{J}$,

$$\xi_i(v) = \sum_{\pi\in\Pi} r_\pi [v(\pi^i \cup i) - v(\pi^i)]$$

$$= \sum_{T\subset N\setminus i} \left(\sum_{\{\pi\in\Pi : \pi^i = T\}} r_\pi \right) [v(T \cup i) - v(T)] \quad .$$

Define, for all $i \in N$ and all $T \subset N\setminus i$,

$$p_T^i = \sum_{\{\pi\in\Pi : \pi^i = T\}} r_\pi \quad ,$$

and let $\phi = (\phi_1,\ldots,\phi_n)$ be the associated collection of probabilistic values. (It is easily verified that, for each $i \in N$, $\{p_T^i : T \subset N\setminus i\}$ is a probability distribution.) Clearly, $\phi = \xi$.

Observe that, for any $v \in \mathcal{J}$,

$$\sum_{i\in N} \xi_i(v) = \sum_{i\in N} \sum_{\pi\in\Pi} r_\pi [v(\pi^i \cup i) - v(\pi^i)]$$

$$= \sum_{\pi\in\Pi} r_\pi \sum_{i\in N} [v(\pi^i \cup i) - v(\pi^i)]$$

$$= \sum_{\pi\in\Pi} r_\pi \cdot v(N) = v(N) \quad .$$

Therefore, since $\phi = \xi$, it follows that ϕ satisfies the efficiency axiom. □

The preceding theorem shows that every random-order value is an efficient probabilistic (group) value. The converse result also holds.

<u>*THEOREM 13*</u>. *Let* $\phi = (\phi_1,\ldots,\phi_n)$ *be a collection of values on* $\mathcal{J}$ *, defined for all* $i \in N$ *and all* $v \in \mathcal{J}$ *by* $\phi_i(v) = \sum_{T \subset N\setminus i} p_T^i[v(T \cup i) - v(T)]$. *Assume that* $\sum_{i \in N} p_{N\setminus i}^i = 1$ *, and that* $\sum_{i \in T} p_{T\setminus i}^i = \sum_{j \notin T} p_T^j$ *for all nonempty* $T \subsetneq N$. *Then there is a random-order value* $\xi = (\xi_1,\ldots,\xi_n)$ *on* $\mathcal{J}$*, such that* $\xi_i(v) = \phi_i(v)$ for all $i \in N$ and $v \in \mathcal{J}$.

<u>Proof</u>. For any $i \in N$ and $T \subset N\setminus i$, define $A^d(T) = \sum_{j \notin T} p_T^j$, and $A(i;T) = p_T^i/A^d(T)$. Consider any ordering $\pi = (i_1,\ldots,i_n) \in \Pi$, and define

$$r_\pi = p_\emptyset^{i_1} \cdot A(i_2;\{i_1\}) \cdot A(i_3;\{i_1,i_2\}) \cdots A(i_n;\{i_1,\ldots,i_{n-1}\}) \quad .$$

It is easily verified, by repeated summation, that

$$\sum_{\pi \in \Pi} r_\pi = \sum_{i_1=1} \sum_{i_2 \notin \{i_1\}} \sum_{i_3 \notin \{i_1,i_2\}} \cdots \sum_{i_n \notin \{i_1,\ldots,i_{n-1}\}} r_{(i_1,\ldots,i_n)} = 1 \quad ,$$

so $\{r_\pi : \pi \in \Pi\}$ is a probability distribution.

Let ξ be the random-order value associated with $\{r_\pi : \pi \in \Pi\}$. Since

$$\xi_i(v) = \sum_{T \subset N\setminus i} \left(\sum_{\{\pi \in \Pi : \pi^i = T\}} r_\pi \right) [v(T \cup i) - v(T)] \quad ,$$

it will suffice to show that for all $i \in N$ and $T \subset N\backslash i$,

$$p_T^i = \sum_{\{\pi:\pi^i=T\}} r_\pi \quad .$$

Observe that

$$\sum_{\{\pi:\pi^i=T\}} r_\pi = \sum_{i_t\in T} \sum_{i_{t-1}\in T\backslash\{i_t\}} \cdots \sum_{i_1\in T\backslash\{i_t,\ldots,i_2\}}$$

$$\sum_{i_{t+2}\notin T\cup\{i\}} \sum_{i_{t+3}\notin T\cup\{i,i_{t+2}\}} \cdots \sum_{i_n\notin T\cup\{i,i_{t+2},\ldots,i_{n-1}\}} r_{(i_1,\ldots,i}$$

$$= \frac{p_T^i}{A^d(T)} \sum_{i_t\in T} \frac{p_{T\backslash\{i_t\}}^{i_t}}{A^d(T\backslash\{i_t\})} \sum_{i_{t-1}\in T\backslash\{i_t\}} \frac{p_{T\backslash\{i_t,i_{t-1}\}}^{i_{t-1}}}{A^d(T\backslash\{i_t,i_{t-1}\})}$$

$$\cdots \sum_{i_1\in T\backslash\{i_t,\ldots,i_2\}} p_\emptyset^{i_1} \sum_{i_{t+2}\notin T\cup\{i\}} A(i_{t+2};T\cup\{i\})$$

$$\cdots \sum_{i_n\notin T\cup\{ i,i_{t+2},\ldots,i_{n-1}\}} A(i_n;T \cup \{i,i_{t+2},\ldots,i_{n-1}\}) \quad .$$

This summation can be carried out explicitly. Proceeding from right to left, the first $n - (t + 1)$ sums each, in turn, have value 1 . Continuing inductively, each sum of the form $\sum_{i_k\in T_k} p_{T_k\backslash i_k}^{i_k}$ is preceded by a factor with denominator $A^d(T_k) = \sum_{j\notin T_k} p_{T_k}^j$. Therefore, from the hypothesis of the theorem, it follows that the expression simplifies to p_T^i , as desired. □

Combining the preceding results, we obtain an interesting observation. A collection of individual probabilistic values is efficient for all games in its domain precisely when the players' probabilistic views of the world are consistent; that is, only when the various $\{p^i_T : T \subset N\backslash i\}$ arise from a single distribution $\{r_\pi : \pi \in \Pi\}$.

9. The Shapley Value

A standard characterization of the Shapley (group) value is as the only value which satisfies the linearity, dummy, symmetry, and efficiency axioms [6] . From our previous results, we can quickly prove the uniqueness of the Shapley value, and simultaneously obtain a simple derivation of the explicit formula for the Shapley value. Traditional proofs center around a consideration of the carrier games in $\mathcal{C}$. It appears that our consideration, as well, of the games in $\hat{\mathcal{C}}$ simplifies matters.

THEOREM 14. Let $\phi = (\phi_1, \ldots, \phi_n)$ *be a group value on* $\mathcal{G}$, $\mathcal{M}$, *or* $\mathcal{S}$. *Assume that each* ϕ_i *satisfies the linearity and dummy axioms, and that* ϕ *satisfies the symmetry and efficiency axioms. Then for every* v *in the domain of* ϕ , *and every* $i \in N$,

$$\phi_i(v) = \sum_{T \subset N\backslash i} \frac{t!(n-t-1)!}{n!} [v(T \cup i) - v(T)] ,$$

where t *generically denotes the cardinality of* T .

Proof. From Theorems 3 and 10, it follows that there is a sequence $\{p_t\}_{t=0}^{n-1}$, such that each $\phi_i(v) = \sum_{T \subset N\backslash i} p_t[v(T \cup i) - v(T)]$. Specializing Theorem 11 to the symmetric case, we must have $\sum_{i \in N} p^i_{N\backslash i} = np_{n-1} = 1$, and

$$\sum_{i \in T} p^{i}_{T \setminus i} = tp_{t-1} = \sum_{j \notin T} p^{j}_{T} = (n - t)p_t \quad \text{for all nonempty} \quad T \subsetneq N \, .$$

Consequently,

$$p_{n-1} = \binom{n-1}{n-1} p_{n-1} = \frac{1}{n} \, ,$$

and

$$\binom{n-1}{t} p_t = \binom{n-1}{t-1} p_{t-1}$$

for all $1 \leqq t \leqq n - 1$. It follows that, for each t , $\binom{n-1}{t} p_t = \frac{1}{n}$, and therefore, $p_t = \frac{t!(n - t - 1)!}{n!}$. □

It may be noted that, upon replacement of the linearity axiom with the transfer axiom, we obtain a similar theorem characterizing the Shapley value on $\mathcal{G}^*$, $\mathcal{M}^*$, or $\mathcal{S}^*$.

10. Interlude

In the preceding sections, we have given two interpretations to a group value $\phi = (\phi_1,\ldots,\phi_n)$ on a collection $\mathcal{J}$ of games. The use of ϕ to indicate an equitable distribution of resources seems reasonable only when ϕ is efficient. However, the interpretation of ϕ as an evaluation function, to be used by a single player comparing various positions within a game, is broadly applicable.

Just as the prospects of various positions can be compared, the prospects of the coalitions in a game can be studied. For any particular game $v \in \mathcal{J}$, we can define a function $\hat{\phi}v : 2^N \to R$, which assigns to each coalition S its total value $\hat{\phi}v(S) = \sum_{i \in S} \phi_i(v)$. Note that this set-function is additive; i.e., if $S \cap T = \emptyset$, then $\hat{\phi}v(S) + \hat{\phi}v(T) = \hat{\phi}v(S \cup T)$.

An important class of games corresponds to economic markets involving a large number of traders, in which each trader holds only a negligible proportion of the total resources of the economy. Such a situation can be conveniently represented by a non-atomic continuum of traders. Since each player is a dummy in the corresponding game, the study of individual values is of little interest. However, the relative prospects of various coalitions (i.e., various segments of the market) can be represented by an additive set-function on the continuum of players. This representation is investigated in the next section. Again, an axiomatic characterization of probabilistic values is our central result.

11. Values of Non-Atomic Games

We now turn to investigate the implications of dropping the efficiency axiom in the context of non-atomic games. The mathematical setting for the study of such games has been spelled out in [11]. For completeness' sake we will quote freely from [11] and first recall several basic definitions and results.

Let $\{I, \mathcal{C}\}$ be a measurable space, isomorphic* to the closed unit interval with its Borel subsets. The term "set function" will mean a mapping v of $\mathcal{C}$ into the reals such that $v(\emptyset) = 0$. In the interpretation, a set function is a game, I is the player set, and $\mathcal{C}$ is the σ-algebra of coalitions. A set function v is monotonic if $S \supset T$ implies $v(S) \geq v(T)$, and is of bounded variation if it is the difference of two monotonic set functions. The collection of all set functions of bounded variation forms a vector space over the reals and will be called BV. FA is the subspace of BV consisting of bounded, finitely additive, signed measures on $\{I, \mathcal{C}\}$ and CA is the set of members of FA that are countably additive.

Let Q be any subspace of BV. The set of all monotonic set functions in Q will be denoted Q^+. A mapping of Q into BV is positive if it maps Q^+ into BV^+.

Let $\mathcal{Q}$ denote the set of all isomorphisms of $\{I, \mathcal{C}\}$ onto

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*Two measurable spaces are called isomorphic if there is a one-to-one function from one onto the other that is measurable in both directions.

itself. Each θ in $\mathcal{A}$ induces a linear mapping θ_* of BV onto itself, defined by $(\theta_* v)(S) = v(\theta(S))$. A subspace Q of BV is symmetric if $\theta_* Q = Q$ for all θ in $\mathcal{A}$.

We are now prepared to define a "value."* Let Q be a symmetric subspace of BV . A value on Q is a positive linear mapping η from Q into** CA such that

(A) For all θ in $\mathcal{A}$ and v in Q , we have

$$\eta(\theta_* v) = \theta_*(\eta v) \quad .$$

(B) For all v in CA $\cap$ Q

$$\eta(v) = v \quad .$$

(A) clearly corresponds to the symmetry axiom for values of finite games. (If we had defined $(\theta_* v)(S) = v(\theta^{-1}(S))$, then (A) would have taken the form $\theta_*(\eta(\theta_* v)) = \eta v$. The correspondence with the symmetry axiom in the finite case would then have been more transparent. However, the terminology used above is more in keeping with [11].) The monotonicity axiom is captured in the requirement that

a value be a positive mapping. (B) has been discussed in [11] (pp. 15-16, pp. 293-4) under the name "Projection Axiom." It may be viewed as the

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*We depart from the usage in [11], where values are required to satisfy the "efficiency axiom," $(\eta v)(I) = v(I)$, instead of (B).

**The effect of considering FA in place of CA , either here or in (B) , is discussed in Remark 3 of Section 12.

non-atomic analogue of the dummy axiom for finite games. Consider a finite additive game $v: 2^N \to \mathbb{R}$, i.e., $v(S \cup T) = v(S) + v(T)$ whenever $S \cap T = \emptyset$. Then each $i \in N$ is a dummy in v, hence $\phi_i(v) = v(i)$, or equivalently $\hat{\phi}(v) = v$.

Our focus in this paper will be on a particular subspace of BV called pNA, which also plays a crucial role in [11]. First let us introduce the variation norm $\|\ \|$ on BV given by

$$\|v\| = \inf\,[u(I) + w(I)] \quad ,$$

where the infimum is taken over all monotonic set functions u and w such that $v = u - w$. BV is a Banach space with this norm (see* Proposition 4.3); FA and CA are closed subspaces of BV (Proposition 4.4). Denote by NA the subspace of CA consisting of non-atomic measures. The subspace pNA is the subset of BV spanned by all powers of measures in NA. The word "spanned" is used here in a topological linear sense; i.e., the space spanned by a subset of BV is the closure (in the variation norm) of the set of all linear combinations of elements of that subset. pNA is clearly closed and symmetric. It is also internal (see Proposition 7.19);

i.e., $\|v\| = \inf\,[u(I) + w(I)]$, where u and w are members of pNA^+ (and not just of BV^+) such that $v = u - w$. A fortiori, pNA is reproducing, i.e., $pNA = pNA^+ - pNA^+$. Therefore, by Proposition 4.15, any positive linear operator from pNA into BV is continuous. In particular, *if* η *is a value on* pNA, *then* η *is continuous or,*

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*All unattributed results are from [11].

equivalently it has finite norm (where the norm of* η *is* $\sup\{||\eta(v)||/||v|| : v \in pNA, \ v \neq 0\}$*).*

Of particular interest are those games v in pNA of the form $v = f \circ \mu$ [i.e., $v(S) = f(\mu(S))$ for all $S \in \mathcal{C}$], where μ is a finite-dimensional vector of measures in NA^+, and f is a real-valued function that is continuously differentiable** on the range of μ, with $f(0) = 0$. (Note that by Lyapunov's theorem the range of μ is compact and convex.) For the proof that $f \circ \mu \in pNA$ for all such μ and f see Proposition 7.1.

We will first focus our attention on games of the form $f \circ \mu$ where μ is one-dimensional. Without loss of generality, $\{I, \mathcal{C}\}$ will henceforth be taken to be the closed unit interval [0,1] with its Borel subsets. λ will stand for the Lebesgue measure on I. Given two measures μ and ξ on I recall that μ is <u>absolutely continuous</u> with respect to ξ, written $\mu << \xi$, if $\mu(S) = 0$ whenever $\xi(S) = 0$. If $\mu << \xi$, then by the Radon-Nikodym theorem there exists a measurable function $f: I \to \mathbb{R}$ such that $\mu(S) = \int_S f d\xi$ for all $S \in \mathcal{C}$. f is called the Radon-Nikodym derivative of μ with respect to ξ and is denoted $d\mu/d\xi$. If it happens that, for some $M < \infty$, $|d\mu/d\xi| < M$ almost everywhere on I, we will say that $d\mu/d\xi$ is <u>bounded</u>.

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*See, for example, Theorem 5.4 of [12] .

**For a precise definition, see page 22 of [11] .

We introduce some more notation:

$$NA_1^+ = \{\mu \in NA^+ : \mu(I) = 1\} .$$

(Thus NA_1^+ is the set of all probability measures on I .)

$$NA_1^+bd = \{\mu \in NA_1^+ : \mu << \lambda , d\mu/d\lambda \text{ is bounded }\} .$$

$$C_0^1 = \{f: I \to \mathbb{R} : f(0) = 0 \text{ and } f \text{ is continuously differentiable}\} .$$

$$C_0^1 NA_1^+ = \{v \in pNA : v = f \circ \mu \text{ where } f \in C_0^1 , \mu \in NA_1^+\} .$$

The following two propositions pave the way towards characterizing all values on pNA .

Proposition I. Let η *be a value on* pNA . *There exists a unique measure* $\xi \in NA_1^+bd$ *such that for any* $\mu \in NA_1^+$ *and any* $f \in C_0^1$

$$(*) \qquad \eta(f \circ \mu) = \left(\int_0^1 f' d\xi\right)\mu ,$$

where f' *denotes the derivative of* f .

Proof. First take μ to be the Lebesgue measure λ . We can show, by exactly the same arguments as in the first part of the proof of Proposition 6.1, that ηv coincides on any two sets of equal λ-measure for any $v \in pNA$, i.e., $(\eta v)(S)$ is a function of $\lambda(S)$ alone. Write

$(\eta(f \circ \lambda))(S) = g_f(\lambda(S))$. Then from $\eta(f \circ \lambda) \in CA$ it follows that g_f is bounded on I , and is additive, i.e., $g_f(x_1 + x_2) = g_f(x_1) + g_f(x_2)$ whenever x_1, x_2 and $x_1 + x_2$ are in $[0,1]$. This implies that $g_f(x) = g_f(1)x$.

Consider the mapping $\Lambda\colon C_0^1 \to \mathbb{R}$ given by:

$$\Lambda(f) = g_f(1) \quad .$$

Note that $g_f(1) = (\eta(f \circ \lambda))(I)$. Since η is linear on pNA , it is clear that Λ is a linear functional. It will be helpful to view Λ with its domain transformed. To this end, let C be the set of all continuous real-valued functions on $[0,1]$. Both C and C_0^1 are vector spaces over the field of real numbers.

Define $d : C_0^1 \to C$ as follows:

$$(d(f))(x) = f'(x) \quad ,$$

for $f \in C_0^1$ and $x \in I$. It can be easily verified that d is a vector space isomorphism,* and that d^{-1} is given by

$$(d^{-1}(\tilde{f}))(x) = \int_0^x \tilde{f}(t)dt \quad ,$$

for $\tilde{f} \in C$ and $x \in I$.

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*That is, d is one-to-one and onto, and linear in both directions.

Now define $\tilde{\Lambda} : C \to \mathbb{R}$ by

$$\tilde{\Lambda}(\tilde{f}) = \Lambda(d^{-1}(\tilde{f})) \quad ,$$

for any $\tilde{f} \in C$. Clearly $\tilde{\Lambda}$ is linear, since Λ and d^{-1} are linear. Moreover $\tilde{\Lambda}$ is positive, i.e., $\tilde{\Lambda}(\tilde{f}) \geqq 0$ whenever $\tilde{f} \geqq 0$. To check this let $\tilde{f} \geqq 0$. Then $f = d^{-1}(\tilde{f})$ is monotonic on I , i.e., $f(x) \geqq f(y)$ whenever $x \geqq y$. Hence, the game $f \circ \lambda$ is a monotonic set function. But since η is a positive mapping, $\eta(f \circ \lambda)$ is also a monotonic function. This implies that $\Lambda(f) = g_f(1) = (\eta(f \circ \lambda))\ (I) \geqq 0$, which easily translates into: $\tilde{\Lambda}(\tilde{f}) \geqq 0$. Hence, $\tilde{\Lambda}$ is a positive linear functional on C.

Then by the Riesz representation theorem* there exists a unique, finite, positive measure ξ on $\{I, \mathcal{C}\}$ such that

$$\tilde{\Lambda}(\tilde{f}) = \int_0^1 \tilde{f} d\xi \quad .$$

This says that

$$\Lambda(f) = \int_0^1 f' d\xi \quad .$$

Recalling that $(\eta(f \circ \lambda))(S) = g_f(1) \cdot \lambda(S)$, we have verified the formula (*) for the case when μ is the Lebesgue measure λ . (However, we have not yet shown that $\xi \in NA_1^+bd$.)

When $\mu \neq \lambda$, let $\hat{\theta}$ be the automorphism of $\{I, \mathcal{C}\}$ such that

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*See, for example, page 34 in [12] .

$\hat{\theta}_* \mu = \lambda$. (Lemma 6.2 assures us that such a $\hat{\theta}$ exists.) Then $\hat{\theta}_*(f \circ \mu) = f \circ (\hat{\theta}_* \mu) = f \circ \lambda$, and so, by what we have just proved, $\eta\hat{\theta}_*(f \circ \mu) = \eta(f \circ \lambda) = (\int_0^1 f' d\xi)\lambda$. Hence, by the symmetry axiom (A),

$$\begin{aligned}
\eta(f \circ \mu) &= \hat{\theta}_*^{-1}\hat{\theta}_*\eta(f \circ \mu) = \hat{\theta}_*^{-1}\eta\hat{\theta}_*(f \circ \mu) \\
&= \hat{\theta}_*^{-1}\left(\left(\int_0^1 f' d\xi\right)\lambda\right) \\
&= \left(\int_0^1 f' d\xi\right)\hat{\theta}_*^{-1}\lambda \\
&= \left(\int_0^1 f' d\xi\right)\mu \quad .
\end{aligned}$$

We shall now proceed to demonstrate that ξ is the type of measure claimed, i.e., $\xi \in NA_1^+ bd$. First suppose, to the contrary, that ξ has atoms. Let $x \in [0,1]$ be any atom of ξ . Construct the sequence* $\{\tilde{f}_n\}_{n \in \mathbb{N}} \subset C$ defined by

$$\tilde{f}_n(y) = \begin{cases} 0 \quad , \quad \text{if } y \in [0, x - \frac{1}{n}] \text{ or if } x \in [x + \frac{1}{n}, 1] \quad ; \\[2ex] \dfrac{y - (x - \frac{1}{n})}{\frac{1}{n}} \quad , \quad \text{if } y \in (x - \frac{1}{n}, x] \cap [0,1] \quad ; \\[2ex] \dfrac{(x + \frac{1}{n}) - y}{\frac{1}{n}} \quad , \quad \text{if } y \in (x, x + \frac{1}{n}) \cap [0,1] \quad . \end{cases}$$

Denote $d^{-1}(\tilde{f}_n)$ by f_n . Then $\{f_n\}_{n \in \mathbb{N}} \subset C_0^1$. Consider the sequence of

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* $\mathbb{N}$ is the set of positive integers.

games $\{v_n\}_{n\in\mathbb{N}}$ in pNA , where $v_n = f_n \circ \lambda$. Clearly v_n is monotonic, hence $||v_n|| = f_n(1) \leqq \frac{1}{n}$. On the other hand, since ηv_n is also monotonic, $||\eta v_n|| = (\eta v_n)(I) = \int_0^1 f'_n d\xi = \int_0^1 \tilde{f}_n d\xi \geqq \xi(x)$. Therefore,

$\frac{||\eta v_n||}{||v_n||} \longrightarrow \infty$, which contradicts the fact that η is continuous on pNA .

We conclude that ξ must be non-atomic.

We now show that $\sup\{\xi(S)/\lambda(S) : S \in \mathcal{C}, \lambda(S) > 0\}$ is finite. If not, there is a sequence of Borel sets $\{S_n\}_{n\in N}$ such that $\xi(S_n)/\lambda(S_n) > n + 1$ and $\lambda(S_n) > 0$, for all n . Since the Lebesgue measure is regular, for any S_n there exists a countable collection of disjoint open intervals, $\{I_j^n\}_{j\in\mathbb{N}}$, such that $Y = \bigcup_{j\in\mathbb{N}} I_j^n \supset S_n$ and $\lambda(Y\backslash S_n) \leqq \lambda(S_n)/n$. Now $\xi(Y)/\lambda(Y) > n$, hence for some j^* we must have $\xi(I_{j^*}^n)/\lambda(I_{j^*}^n) > n$. Let $I_{j^*}^n = (\alpha^n, \beta^n)$, and put $\hat{\alpha}^n = \max\{0, \alpha^n - (\frac{\beta^n - \alpha^n}{n})\}$, $\hat{\beta}^n = \min\{1, \beta^n + (\frac{\beta^n - \alpha^n}{n})\}$.

Define $\{\tilde{g}_n\}_{n\in\mathbb{N}} \subset C$ as follows:

$$\tilde{g}_n(y) = \begin{cases} 1 & \text{if } y \in [\alpha^n, \beta^n] , \\ \dfrac{y - \hat{\alpha}^n}{\alpha^n - \hat{\alpha}^n} & \text{if } y \in [\hat{\alpha}^n, \alpha^n) , \\ \dfrac{\hat{\beta}^n - y}{\hat{\beta}^n - \beta^n} & \text{if } y \in (\beta^n, \hat{\beta}^n] , \\ 0 & \text{otherwise} . \end{cases}$$

Again consider the sequence $\{v_n\}_{n\in\mathbb{N}}$ in pNA given by $v_n = g_n \circ \lambda$ where $g_n = d^{-1}(\tilde{g}_n)$. Arguing as before we derive:

$$||v_n|| = g_n(1) = \int_0^1 \tilde{g}_n d\lambda \leq \lambda(I^n_{j*}) + \frac{\beta^n - \alpha^n}{n} \quad ;$$

$$||\eta v_n|| = \int_0^1 \tilde{g}_n d\xi \geq \xi(I^n_{j*}) \ . \quad \text{But then,}$$

$$\frac{||\eta v_n||}{||v_n||} \geq \frac{\xi(I^n_{j*})}{\lambda(I^n_{j*}) + \left(\frac{\beta^n - \alpha^n}{n}\right)} = \frac{\xi(I^n_{j*})}{[\lambda(I^n_{j*})](1 + \frac{1}{n})} \geq \frac{n^2}{n+1} \longrightarrow \infty \ ,$$

a contradiction. This proves that there exists an M , $0 < M < \infty$, such that $\sup \{\xi(S)/\lambda(S) : S \in \mathcal{C}, \lambda(S) > 0\} < M$. Therefore, for any $\varepsilon > 0$ and any $S \in \mathcal{C}$, $\lambda(S) < \varepsilon/M$ implies $\xi(S) < \varepsilon$. Then, $\xi << \lambda$, and therefore $d\xi/d\lambda$ exists. Clearly, $d\xi/d\lambda \geq 0$ almost everywhere. We assert that also $d\xi/d\lambda \leq M$ almost everywhere. If not, let $T = \{t \in I : (d\xi/d\lambda)(t) > M\}$, with $\lambda(T) > 0$. Then, $\xi(T) = \int_T (d\xi/d\lambda)d\lambda > M\lambda(T)$, a contradiction. This shows that $d\xi/d\lambda$ is bounded.

Finally, it remains to show that $\xi(I) = 1$. Take the game $f \circ \lambda$ where $f(x) = x$ for $x \in I$. Then, by (B), we must have $\eta(f \circ \lambda) = \lambda$. On the other hand, we have shown that $\eta(f \circ \lambda) = (\int_0^1 f' d\xi)\lambda = (\int_0^1 d\xi)\lambda$ $= (\xi(I))\cdot\lambda$. Hence $\xi(I) = 1$. □

Proposition II. *For each* $\xi \in NA_1^+bd$ *, there is a unique value* η_ξ *on* pNA *which admits of the representation below. Let* v *in* pNA *be such that there exist* μ , f , *and* U *as follows:*

(i) μ *is a finite dimensional vector of non-atomic measures with range* H , f *is a real-valued function on* H *and continuously differentiable there with* $f(0) = 0$, U *is a compact convex neighborhood in* H *of the diagonal* $[0,\mu(I)]$ *, and*

$$v(S) = f(\mu(S)) \quad \textit{whenever} \quad \mu(S) \in U .$$

Then, for all $S \in \mathcal{C}$,

$$\textit{(ii)} \quad (\eta_\xi v)(S) = \int_0^1 f_{\mu(S)}(t\mu(I))d\xi(t)$$

where $f_{\mu(S)}$ *is the derivative of* f *in the direction* $\mu(S)$.

Proof. Fix $\xi \in NA_1^+bd$, and let v, μ , f , U be as in (i). Define the signed measure $\nu_{f,\mu}$ by

$$\nu_{f,\mu}(S) = \int_0^1 f_{\mu(S)}(t\mu(I))d\xi(t) \quad .$$

It is easy to verify the countable additivity of $\nu_{f,\mu}$ from the explicit formula. This is carried through for the case $\xi = \lambda$ in the beginning of the proof of Proposition 7.6. An exactly analogous argument can be used when $\xi \neq \lambda$.

Let $I = S^+ \cup S^-$ be a Hahn decomposition (see Theorem 6.14 in [12]) of I with respect to $\nu_{f,\mu}$; i.e., $\nu_{f,\mu}$ is nonnegative on S^+ and its subsets, nonpositive on S^- and its subsets, and $S^+ \cap S^- = \emptyset$. Put $y = \mu(S^+)$ and $b = \mu(I)$, so $\mu(S^-) = b - y$. Then as shown** in the proof of Proposition 7.6,

$$||v|| \geq \int_0^1 |f_y(tb)|dt + \int_0^1 |f_{b-y}(tb)|dt \quad .$$

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*When $\mu(S) = 0$ we define the integral to be 0 .

**In [11] it is proved that $||v|| \geq \left|\int_0^1 f_y \;(tb)\; dt\right| + \left|\int_0^1 f_{b-y}(tb)dt\right|$.

But their proof in fact shows the stronger inequality we have used.

But then, choosing M so that $d\xi/d\lambda < M$ almost everywhere, we get

$$\begin{aligned}
\text{(iii)}\quad M\|v\| &\geq M\int_0^1 |f_y(tb)|\,dt + M\int_0^1 |f_{b-y}(tb)|\,dt \\
&\geq \int_0^1 |f_y(tb)|\,d\xi(t) + \int_0^1 |f_{b-y}(tb)|\,d\xi(t) \\
&\geq \left|\int_0^1 f_y(tb)\,d\xi(t)\right| + \left|\int_0^1 f_{b-y}(tb)\,d\xi(t)\right| \\
&= |\nu_{f,\mu}(S^+)| + |\nu_{f,\mu}(S^-)| \\
&= \|\nu_{f,\mu}\| \;.
\end{aligned}$$

Let D be the linear subspace* of set functions v in pNA that can be represented by some μ, f, and U as in (i). For every such v, μ, f, U define

$$\phi v = \nu_{f,\mu} \;.$$

We need to check that this is an admissible definition, i.e., that it does not depend on the choice of μ, f, and U . Indeed, this too has been done in Proposition 7.6 for the case $\xi = \lambda$. When $\xi \neq \lambda$ no difficulty arises and the same proof may be invoked.

D contains all the linear combinations of powers of measures in NA, so it is dense in pNA. (iii) shows that $\|\phi(v)\|/\|v\| \leq M$ for all $v \in D$. Furthermore ϕ maps D into CA, which is complete. Therefore, there is a unique extension of ϕ to a continuous linear operator

*This subspace is named Q in [11].

from pNA to CA . Call this extended operator η_ξ . It is the only operator on pNA which satisfies (ii) for all v as in (i), i.e., all v in D .

To conclude the proof, we must verify that η_ξ is a value; i.e., η_ξ is positive and satisfies (A) and (B) . To check (A) (symmetry), consider $v = f \circ \mu \in C_0^1 NA_1^+$. Then for each $\theta \in \mathcal{G}$, $\theta_* \mu \in NA_1^+$, hence $\theta_* v = f \circ (\theta_* \mu) \in C_0^1 NA_1^+$. Also

$$\begin{aligned} \eta_\xi \theta_* v &= \left(\int_0^1 f' d\xi \right) \theta_* \mu \\ &= \theta_* \left(\int_0^1 f' d\xi \right) \mu \\ &= \theta_* \eta_\xi v \quad . \end{aligned}$$

Since both η_ξ and θ_* are continuous on pNA , it follows that $\eta_\xi \theta_* - \theta_* \eta_\xi$ is a continuous linear operator on pNA that vanishes on $C_0^1 NA_1^+$. Therefore, it vanishes on the (topological linear) span of $C_0^1 NA_1^+$, which is pNA . This proves symmetry.

For any $Q \subset BV$, let us denote $\bar{Q}$ the closure of Q in the variation norm. Let P be the space of all polynomials in non-atomic measures. Whenever we write $v = f \circ \mu \in P$, we take f to be a polynomial and μ a vector of measures in NA . Clearly, $\bar{P} = pNA$.

An alternative definition of the variation norm will be useful in the sequel. Let Ω be a nested chain of sets $\emptyset = S_0 \subset S_1 \subset \ldots \subset S_m = I$, and let $v \in BV$. Put $||v||_\Omega = \sum_{i=1}^{m} |v(S_i) - v(S_{i-1})|$. Then

Proposition 4.1 establishes that $||v|| = \sup ||v||_\Omega$, where the supremum is taken over all chains Ω .

To check that η_ξ satisfies (B), the projection axiom, first consider $v = f \circ \mu \in CA \cap P$. Without loss of generality (see page 42 of [11]), we can assume that H , the range of μ , is full-dimensional.

We assert that f must be linear. Pick any $x \in H$ and consider the ray $R_x = \{tx : t \geq 0 , tx \in H\}$. Let t_1x and t_2x be in R_x such that $(t_1 + t_2)x \in R_x$. Since $R_x \subset H$, there is an $S \in \mathcal{C}$ such that $\mu(S) = (t_1 + t_2)x$. By Lyapunov's theorem (applied to μ on S), for some $T \subset S$, $\mu(T) = t_1x$. Hence $\mu(S\backslash T) = t_2x$. But $f \circ \mu \in CA$, and therefore

$$\begin{aligned} f((t_1 + t_2)x) &= (f \circ \mu)(S) \\ &= (f \circ \mu)(T) + (f \circ \mu)(S\backslash T) \\ &= f(t_1x) + f(t_2x) \quad . \end{aligned}$$

Thus f is additive on R_x ; clearly it is bounded. Consequently f is linear on R_x for any $x \in H$; i.e., f is homogeneous of degree one on H . Now consider ∇f , the gradient of f . Due to homogeneity, ∇f is constant on each ray R_x . Since these rays all contain the origin, ∇f in fact must be constant throughout H . It follows that $f(x) = c \cdot x$ for all $x \in H$, where c is this constant gradient vector.

Now,

$$\begin{aligned} (\eta_\xi v)(S) &= \int_0^1 f_{\mu(S)}(t\mu(I))d\xi(t) \\ &= c \cdot \mu(S) \int_0^1 d\xi(t) \\ &= f(\mu(S)) \\ &= v(S) \quad . \end{aligned}$$

Thus, $\eta_\xi v = v$ for all $v \in CA \cap P$.

To complete our verification of (B), it will suffice to show that $CA \cap P = CA \cap pNA$. Since $CA \cap P = NA$ and $\overline{P} = pNA$, we need only show that any limit of non-atomic set functions is non-atomic. Suppose, to the contrary, that some $v \in pNA$ has an atom x in I , i.e., for some $S \subset I \backslash \{x\}$, $v(S \cup \{x\}) \neq v(S)$. Take $\{v_n\}_{n \in \mathbf{N}} \subset P$ such that $v_n \to v$. Clearly each v_n is non-atomic, so $v_n(S \cup \{x\}) = v_n(S)$ for all n . But then, considering the chain $\{\emptyset, S, S \cup \{x\}, I\}$, we get

$$||v_n - v|| \geq |(v_n - v)(S \cup \{x\}) - (v_n - v)(S)|$$
$$= |v(S) - v(S \cup \{x\})| \quad ,$$

i.e., $||v_n - v|| \not\longrightarrow 0$, which is a contradiction.

Finally, we must show that η_ξ is positive on pNA . We will begin by showing this on D , i.e., $\eta_\xi(D^+) \subset CA^+$. Suppose $v \in D^+$ and $(\eta_\xi v)(S) < 0$ for some $S \in \mathcal{C}$. Let $v(T) = (f \circ \mu)(T)$ whenever $\mu(T) \in U$, where μ , f , U are as in (i) . Then since

$$(\eta_\xi v)(S) = \int_0^1 f_{\mu(S)}(t\mu(I))d\xi(t) \quad ,$$

$f_{\mu(S)}(t\mu(I))$ must be negative for all t in some subset of $[0,1]$ of positive ξ-measure . Select a particular such t , satisfying $0 < t < 1$. (Such a t always exists because ξ is non-atomic.) It can be shown, using Lyapunov's theorem, that for <u>any</u> $0 < \tau < 1 - t$ there exist two <u>disjoint</u> sets R_τ and T_τ in $\mathcal{C}$ such that

$$\mu(R_\tau) = t\mu(I)$$

$$\mu(T_\tau) = \tau\mu(S) \quad .$$

(For a proof of this see Note 2 of Section 7 in [11].) Put $P_\tau = R_\tau \cup T_\tau$. For τ sufficiently close to 0 , we have $\mu(R_\tau)$ and $\mu(P_\tau)$ in U , and

$$\begin{aligned} v(P_\tau) = f(\mu(P_\tau)) &= f(t\mu(I) + \tau\mu(S)) \\ &\approx f(t\mu(I)) + \tau f_{\mu(S)}(t\mu(I)) \\ &< f(t\mu(I)) \\ &= f(\mu(R_\tau)) \\ &= v(R_\tau) \quad , \end{aligned}$$

contradicting that $v \in D^+$.

It is straightforward to show that if $||v_n - v|| \longrightarrow 0$ and if v_n is monotonic for every n , then v is monotonic. This implies that: (a) CA^+ is closed since CA is closed; (b) $\overline{D^+} \subset (\overline{D})^+$. Since η_ξ is continuous, we immediately get $\eta_\xi(\overline{D^+}) \subset CA^+$ from $\eta_\xi(D^+) \subset CA^+$. We wish to show that $\eta_\xi(\overline{D})^+) \subset CA^+$. (Note $\overline{D} = pNA$.)

For each $k > 0$ and each m with $1 \leq m \leq 2^k$, define a measure λ_m^k by

$$\lambda_m^k(S) = 2^k \lambda\left(S \cap \left[\frac{m-1}{2^k}, \frac{m}{2^k}\right)\right) \quad ,$$

where λ is Lebesgue measure. Let λ^k be a vector measure, of dimension 2^k , defined by

$$\lambda^k = (\lambda_1^k, \ldots, \lambda_{2^k}^k) \quad ;$$

the range of λ^k is the closed unit cube $R_k = [0,1]^{2^k}$. Denote by A the set of all set functions of the form $f \circ \lambda^k$, where $k > 0$, *and f is continuously differentiable on* R_k and takes the value 0 at the origin. Obviously, $A \subset D$.

Let $v \in (\bar{A})^+$. Then there is a sequence $\{v_n\}_{n \in \mathbb{N}} \subset A$ such that $||v - v_n|| \longrightarrow 0$. Since A is internal (Lemma 7.18), we can write $v_n = u_n - w_n$, where $u_n, w_n \in A^+$ and $||v_n|| - (u_n(I) + w_n(I)) \longrightarrow 0$. Consider $||v - v_n|| = ||v - (u_n - w_n)||$. This bounds the variation of $v - u_n + w_n$ on the chain $\{\emptyset, I\}$; hence, $v(I) - u_n(I) + w_n(I) \longrightarrow 0$. Subtracting these two limiting equations, we obtain $(||v_n|| - v(I)) - 2w_n(I) \to 0$. But $||v_n|| \to ||v|| = v(I)$, so $w_n(I) \to 0$. Therefore, $||v - u_n|| \longrightarrow 0$, and $v \in \overline{A^+}$.

Clearly $\overline{A^+} \subset (\bar{A})^+$, and we have just shown that $(\bar{A})^+ \subset \overline{A^+}$. Hence, $(\bar{A})^+ = \overline{A^+}$. Since $A \subset D$, we have $\overline{A^+} \subset \overline{D^+}$. Consequently (recalling that $\eta_\xi(\overline{D^+}) \subset CA^+$), $\eta_\xi((\bar{A})^+) \subset CA^+$. Take any $v \in (\bar{D})^+$. As shown in the proof of Proposition 7.19 in [11], there is an automorphism θ of $\{I, \mathcal{C}\}$ such that $\theta_* v \in (\bar{A})^+$. But then $\eta_\xi(v) = \theta_*^{-1}\eta_\xi(\theta_* v) \in \theta_*^{-1} CA^+ = CA^+$. This proves that $\eta_\xi((\bar{D})^+) \subset CA^+$, i.e., that η_ξ is positive on pNA. □

At last we come to the main theorem in this section.

<u>THEOREM 15</u>. *The set of values on* pNA *is the set* $\{\eta_\xi : \xi \in NA_1^+ bd\}$, *where* η_ξ *is as in Proposition II*.

<u>Proof</u>. We verified in Proposition II that every η_ξ is a value on pNA. Now suppose ϕ is a value on pNA. By Proposition I there exists a unique $\xi \in NA_1^+ bd$ such that

$$\phi(f \circ \mu) = \left(\int_0^1 f' d\xi\right)\mu$$

for any $f \circ \mu \in C_0^1 NA_1^+$. ϕ and η_ξ coincide on $C_0^1 NA_1^+$, which spans pNA . Since ϕ is continuous on pNA , and since CA is closed, there is a unique linear extension of ϕ from the domain $C_0^1 NA_1^+$ to the domain spanned by $C_0^1 NA_1^+$; i.e., to pNA . That this extension coincides with η_ξ is obvious. □

12. Remarks

(1) The non-atomic values $\{\eta_\xi | \xi \in NA_1^+ bd\}$ derived axiomatically on pNA can also be obtained from aymptotic considerations. This reveals the sense in which they approximate values of finite games, and was carried out by one of us in [3]. The results in [3] help to justify the term "probabilistic value" for η_ξ . It is shown there that for any $v \in pNA$, and any sequence $\{v_n\}_{n \in \mathbb{N}}$ of finite games that "converges" to v in an appropriate sense, we have

$$\hat{\phi}_\xi v_n \longrightarrow \eta_\xi v \ ,$$

where v_n is a game with $m = m(n)$ players, and $\hat{\phi}_\xi v_n$ is the additive set-function derived from the symmetric probabilistic value with $p_t(\xi) = \int_0^1 s^t (1 - s)^{m-1-t} d\xi(s)$ for $0 \leq t \leq m - 1$. In this sense, η_ξ is a probabilistic value for a game with a continuum of players.

(2) Although the probabilistic values η_ξ are in general not efficient, it is worth noting that for non-atomic market games* (with transferable utilities) they do all become efficient; indeed they coincide with each other, and in particular with the Shapley value η_λ, which is always efficient. To see this, first consider a market game v of finite type. As shown in [3], v can be approximated by a sequence $v_n = {}_nf \circ \mu_n \in D$, where each ${}_nf$ is homogeneous. But $\eta_\xi v = \lim \eta_\xi v_n$. Since ${}_nf$ is homogeneous, ${}_nf_{\mu(S)}$ is constant along the diagonal $[0,\mu(I)]$. Therefore, by formula (ii) of Proposition II, $\eta_\xi v_n$ is independent of $\xi \in NA_1^+bd$, and then so is $\eta_\xi v$. If the market game v is not of finite type, we may view it as a limit of market games of finite type (with the number of types increasing to infinity) as in [11]. Again a limit argument shows that η_ξ is independent of ξ. By Theorem J, the upshot of this is that *all the* η_ξ *-values are in the core of any market game, and coincide with the competitive payoff of the market.* Even in the case of non-transferable-utility markets, we may define " η_ξ -value allocations" exactly as the (Shapley) value allocations were defined in [13]. Again it follows from the results in [13] and our preceding arguments that η_ξ allocations, for any $\xi \in NA_1^+bd$, coincide with the competitive allocations of the market.

(3) There are two ways in which FA could have been used in place of CA : as the range of the value operator or in the projection axiom (B). If FA were used for the range, leaving (B) unchanged, our results would be unaffected; it is easy to deduce that the effective range would still be just CA. We think that the projection axiom, on all of FA, is a consequence of Proposition II. However, we have no proof of this as yet.

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*For their definition see Chapter 6 of [11] .

(4) Earlier in this paper it was shown, for the finite-player case that the Shapley value is characterized uniquely by the linearity, dummy, symmetry, and efficiency axioms. For the non-atomic case, we can obtain a similar result. The natural non-atomic analogue of the efficiency axiom is

(C) $\qquad (\eta v)(I) = v(I)$, for all $v \in Q$.

Of the values on pNA considered in Proposition II only η_λ satisfies this additional axiom.

To see this, recall that for any value η_ξ ,

$$(\eta_\xi(f \circ \mu))(S) = \int_0^1 f_{\mu(S)}(t\mu(I))d\xi(t)$$

for $f \circ \mu \in C_0^1 NA_1^+ \subset D$. In particular, for any $[\alpha,\beta] \subset I$, consider $v_n = g_n \circ \lambda$, where g_n is defined as in the proof of Proposition I (with $\alpha^n = \alpha$, $\beta^n = \beta$; also set $\hat{\alpha} = \hat{\alpha}^n$, $\hat{\beta} = \hat{\beta}^n$) . From (C) we must have

$$(\eta_\xi v_n)(I) = \int_0^1 g_n'(t)d\xi(t) = (g_n \circ \lambda)(I) \quad .$$

Now

$$\beta - \alpha \leq (g_n \circ \lambda)(I) \leq (\beta - \alpha)(1 + \frac{1}{n}) ,$$

and

$$\xi([\alpha,\beta]) \leqq \int_0^1 g_n'(t)\,d\xi(t) \leqq \xi([\hat{\alpha},\hat{\beta}]) \leqq \xi([\alpha,\beta]) + M \cdot \frac{2}{n}(\beta - \alpha) \quad ,$$

where M bounds $d\xi/d\lambda$. Letting $n \to \infty$ in the preceding inequalities, and noting that the two central terms are equal, we obtain $\xi([\alpha,\beta]) = \beta - \alpha$ for all $[\alpha,\beta] \subset I$; i.e., $\xi = \lambda$.

Indeed, we could have replaced (B) with (C) in our original definition of "value." Propositions I and II would then have held with " λ " replacing " $\xi \in NA_1^+ bd$ "; their proofs would have required only minor modifications. With the exception of the argument requiring the Reisz representation theorem, this would amount to the approach of [11].

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